Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 200 (2007) 701 – 704
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Explicit solution of the operator equation $A^*X + X^*A = B$ ☆

Dragan S. Djordjević*

Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, Višegradska 33,18000 Niš, Serbia, Serbia and Montenegro

Received 7 August 2005; received in revised form 15 November 2005

Abstract

In this paper we find the explicit solution of the equation

$$A^*X + X^*A = B$$

for linear bounded operators on Hilbert spaces, where X is the unknown operator. This solution is expressed in terms of the Moore-Penrose inverse of the operator A. Thus, results of J. H. Hodges [Some matrix equations over a finite field, Ann. Mat. Pura Appl. 44 (1957) 245–550] are extended to the infinite dimensional settings.

MSC: 47A62; 47A52; 15A24

Keywords: Operator equations; Moore–Penrose inverse

1. Introduction

In this paper H and K denote arbitrary Hilbert spaces. We use $\mathscr{L}(H, K)$ to denote the set of all linear bounded operators from H to K. Also, $\mathscr{L}(H) = \mathscr{L}(H, H)$.

For given operators $A \in \mathscr{L}(H, K)$ and $B \in \mathscr{L}(H)$, we are interested in finding the solution $X \in \mathscr{L}(H, K)$ of the equation

$$A^*X + X^*A = B. \tag{1}$$

This equation is considered for matrices over a finite field (see [7]).

We mention similar matrix equations, which have applications in control theory. These equations are investigated for matrices over fields, mostly **R** or **C**. The equation $CX - XA^\top = B$ is the Sylvester equation [8]. More general equation $AX - XF = BY$ is considered in [10]. One special and important case is the Lyapunov equation $AX + XA^\top = B$ [9]. Also, the generalized Sylvester equation $AV + BW = EVJ + R$ with unknown matrices V and W, has many applications in linear systems theory (see [4]).

Present paper deals with the extension of results from [7] to infinite dimensional settings.

☆ Supported by Grant No. 144003 of the Ministry of Science, Republic of Serbia.
* Tel.:/fax: +381 18 533 014.
E-mail addresses: dragan@pmf.ni.ac.yu, ganedj@Eunet.ni.ac.yu (D.S. Djordjević).

0377-0427/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2006.01.023

For $A \in \mathscr{L}(H, K)$ we use $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively, to denote the range and the null-space of A. The Moore–Penrose inverse of A, denoted by $A^\dagger$, is the unique operator $A \in \mathscr{L}(K, H)$ satisfying the following conditions:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists if and only if $\mathscr{R}(A)$ is closed. For properties and applications of the Moore–Penrose inverse see [1,3,2,5].

Let $A \in \mathscr{L}(H, K)$ have a closed range. Then $AA^\dagger$ is the orthogonal projection from K onto $\mathscr{R}(A)$ (parallel to $\mathscr{N}(A^\dagger) = \mathscr{N}(A^*)$) and $A^\dagger A$ is the orthogonal projection from H onto $\mathscr{R}(A^\dagger) = \mathscr{R}(A^*)$ (parallel to $\mathscr{N}(A)$). It follows that A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Now, the operator $A^\dagger$ has the following form:

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}.$$

Using these matrix forms of operators with closed ranges and properties of the Moore–Penrose inverse, we solve Eq. (1).

2. Results

First, we solve Eq. (1) in the case when A is invertible. It can easily be seen that the proof of the following Theorem 2.1 is valid in rings with involution.

Theorem 2.1. *Let $A \in \mathscr{L}(H, K)$ be invertible and $B \in \mathscr{L}(H)$. Then the following statements are equivalent*:

(a) *There exists a solution $X \in \mathscr{L}(H, K)$ of Eq.* (1).
(b) $B = B^*$.

If (a) *or* (b) *is satisfied*, *then any solution of Eq.* (1) *has the form*

$$X = \tfrac{1}{2}(A^*)^{-1}B + ZA, \tag{2}$$

where $Z \in \mathscr{L}(K)$ satisfy $Z^* = -Z$.

Proof. (a) $\to$ (b): Obvious.

(b) $\to$ (a): It is easy to see that any operator X of the form (2) is a solution of Eq. (1). On the other hand, let X be any solution of (1). Then $X = (A^*)^{-1}B - (A^*)^{-1}X^*A$ and $(A^*)^{-1}X^* = (A^*)^{-1}BA^{-1} - XA^{-1}$. We have

$$\begin{aligned} X &= \tfrac{1}{2}(A^*)^{-1}B + (\tfrac{1}{2}(A^*)^{-1}BA^{-1} - (A^*)^{-1}X^*)A \\ &= \tfrac{1}{2}(A^*)^{-1}B + (\tfrac{1}{2}[(A^*)^{-1}X^* + XA^{-1}] - (A^*)^{-1}X^*)A \\ &= \tfrac{1}{2}(A^*)^{-1}B + \tfrac{1}{2}(XA^{-1} - (A^*)^{-1}X^*)A. \end{aligned}$$

Taking $Z = \frac{1}{2}(XA^{-1} - (A^*)^{-1}X^*)$, we get $Z^* = -Z$. $\square$

Now, we solve Eq. (1) in the case when A has a closed range.

Theorem 2.2. *Let $A \in \mathscr{L}(H, K)$ have a closed range and $B \in \mathscr{L}(H)$. Then the following statements are equivalent*:

(a) *There exists a solution $X \in \mathscr{L}(H, K)$ of Eq.* (1).

(b) $B = B^*$ *and* $(I - A^\dagger A)B(I - A^\dagger A) = 0$.

If (a) *or* (b) *is satisfied, then any solution of Eq.* (1) *has the form*

$$X = \tfrac{1}{2}(A^*)^\dagger BA^\dagger A + (A^*)^\dagger B(I - A^\dagger A) + (I - AA^\dagger)Y + AA^\dagger ZA, \tag{3}$$

where $Z \in \mathscr{L}(K)$ *satisfies* $A^*(Z + Z^*)A = 0$, *and* $Y \in \mathscr{L}(H, K)$ *is arbitrary.*

Proof. (a) → (b): Obviously, $B^* = B$. Also,

$$\begin{aligned}(I - A^\dagger A)B(I - A^\dagger A) &= (I - A^\dagger A)(A^*X + X^*A)(I - A^\dagger A)\\ &= (A^* - (AA^\dagger A)^*)X(I - A^\dagger A) + (I - A^\dagger A)X^*A(I - A^\dagger A) = 0.\end{aligned}$$

(b) → (a): Note that the condition $(I - A^\dagger A)B(I - A^\dagger A) = 0$ is equivalent to $B = A^\dagger AB + BA^\dagger A - A^\dagger ABA^\dagger A$. Any operator X of the form (3) is a solution of Eq. (1).

On the other hand, suppose that X is a solution of Eq. (1). Since $\mathscr{R}(A)$ is closed, we have $H = \mathscr{R}(A^*) \oplus \mathscr{N}(A)$ and $K = \mathscr{R}(A) \oplus \mathscr{N}(A^*)$. Now, A has the matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Conditions $B = B^*$ and $(I - A^\dagger A)B(I - A^\dagger A) = 0$ imply that B has the form

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} R(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

where $B_1^* = B_1$. Let X have the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix}.$$

Then $A^*X + X^*A = B$ implies $A_1^*X_{11} + X_{11}^*A_1 = B_1$ and $A_1^*X_{12} = B_2$. Hence, $X_{12} = (A_1^*)^{-1}B_2$. Since A_1 is invertible, from Theorem 2.1 it follows that X_{11} has the form $X_{11} = \frac{1}{2}(A_1^*)^{-1}B_1 + Z_1A_1$, for some operator $Z_1 \in \mathscr{L}(\mathscr{R}(A))$ satisfying $Z_1^* = -Z_1$. Hence,

$$X = \begin{bmatrix} \frac{1}{2}(A_1^*)^{-1}B_1 + Z_1A_1 & (A_1^*)^{-1}B_2 \\ X_{21} & X_{22} \end{bmatrix},$$

X_{21} and X_{22} can be taken arbitrary. Let

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix}$$

and

$$Z = \begin{bmatrix} Z_1 & Z_{12} \\ -Z_{12}^* & Z_{22} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

where Y_{11}, Y_{12} and Z_2 are arbitrary. Note that $A^*(Z + Z^*)A = 0$.

Then

$$\tfrac{1}{2}(A^*)^\dagger BA^\dagger A = \begin{bmatrix} \frac{1}{2}(A_1)^{-1}B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (A^*)^\dagger B(I - A^\dagger A) = \begin{bmatrix} 0 & (A_1^*)^{-1}B_2 \\ 0 & 0 \end{bmatrix},$$
$$(I - AA^\dagger)Y = \begin{bmatrix} 0 & 0 \\ X_{21} & X_{22} \end{bmatrix}$$

and

$$AA^\dagger ZA = \begin{bmatrix} Z_1A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, X has the form (3). □

It is a consequence of the Gelfand-Naimark-Segal theorem and the Harte–Mbekhta theorem [6] that Theorem 2.2 holds in C^*-algebras also.

By exactly similar arguments, we obtain the following analogue of Theorem 2.2, in which Eq. (1) is replaced by

$$A^*X - X^*A = B. \tag{4}$$

Theorem 2.3. *Let $A \in \mathscr{L}(H, K)$ have a closed range and $B \in \mathscr{L}(H)$. Then the following statements are equivalent*:

(a) *There exists a solution $X \in \mathscr{L}(H, K)$ of Eq.* (4).
(b) $B = -B^*$ *and* $(I - A^\dagger A)B(I - A^\dagger A) = 0$.

If (a) *or* (b) *is satisfied*, *then any solution of Eq.* (4) *has the form*

$$X = \tfrac{1}{2}(A^*)^\dagger BA^\dagger A + (A^*)^\dagger B(I - A^\dagger A) + (I - AA^\dagger)Y + AA^\dagger ZA, \tag{5}$$

where $Z \in \mathscr{L}(K)$ satisfies $A^*(Z - Z^*)A = 0$, and $Y \in \mathscr{L}(H, K)$ is arbitrary.

Acknowledgement

I am grateful to both the referees for helpful comments concerning the paper.

References

[1] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer, Berlin, 2003.
[2] S.L. Campbell, C.D. Meyer Jr., Generalized Inverses of Linear Transformations, Dover Publications, New York, 1991.
[3] S.R. Caradus, Generalized Inverses and Operator Theory, Queen's Paper in Pure and Applied Mathematics, Queen's University, Kingston, Ontario, 1978.
[4] G.R. Duan, The solution to the matrix equation $AV + BW = EVJ + R$, Appl. Math. Lett. 17 (2004) 1197–1202.
[5] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Marcel Dekker, New York, 1988.
[6] R.E. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103 (1992) 71–77.
[7] J.H. Hodges, Some matrix equations over a finite field, Ann. Mat. Pura Appl. IV. Ser. 44 (1957) 245–250.
[8] P. Kirrinnis, Fast algorithms for the Sylvester equation $AX - XB^\top = C$, Theoret. Comput. Sci. 259 (2001) 623–638.
[9] D.C. Sorensen, A.C. Antoulas, The Sylvester equation and approximate balanced reduction, Linear Algebra Appl. 351–352 (2002) 671–700.
[10] B. Zhou, G.R. Duan, An explicit solution to the matrix equation $AX - XF = BY$, Linear Algebra Appl. 402 (2005) 345–366.